LOGARITHMIC SOLUTIONS OF BIANCHI'S EQUATION

By H. Bateman

California Institute of Technology

Communicated July 20, 1933

The partial differential equation

$$\frac{\partial^n V}{\partial x_1 \partial x_2 \ldots \partial x_n} = MV$$

was discussed by Bianchi[1] with the aid of the methods of Riemann and Picard. The results were extended to a more general equation which was also studied by Niccoletti.[2] The original equation, for a constant value of M, was studied later by Sibirani[3] in connection with a generalization of the Bessel function and some partial differential equations were listed which could be solved with the aid of this function. The case in which M is constant has also been studied by Chaundy[4] who gives some solutions in the form of definite integrals which we wish to obtain here with the aid of Murphy's theorem.

Putting $M = 1$ we write $z = l_1 + l_2 + \ldots l_n$, where $l_1 = \log x_1$, $l_2 = \log x_2, \ldots l_n = \log x_n$. The differential equation is then satisfied by $V = F(z)$ if

$$D^n F = e^z F \qquad (1)$$

where D denotes the operator d/dz. The general solution of (1) may be written in the form[5]

$$F = \sum_{s=0}^{n-1} \frac{z^s}{s!} U_s(z) \qquad (2)$$

where $U_0, U_1, U_2, \ldots U_{s-1}$ are functions which satisfy the equations

$$(D + E)^n U_{n-s} = e^z U_{n-s} \quad (s = 1, 2, \ldots n) \qquad (3)$$

E being an operator such that $EU_{n-p} = U_{n-p+1}$ and U_s being defined to be zero when $s \geqq n$. The equation for U_{n-1} is

$$D^n U_{n-1} = e^z U_{n-1} \qquad (4)$$

and we may write

$$U_{n-1} = C_{n-1}\left[1 + \frac{e^z}{(1!)^n} + \frac{e^{2z}}{(2!)^n} + \ldots\right] = c_{n-1}E_n(e^z) \qquad (5)$$

where c_{n-1} is an arbitrary constant and $E_n(x)$ is Chaundy's notation for the function which Sibirani denotes by the symbol $L_{0,n}(nx)$, his definition of $L_{\nu,k}(x)$ being as follows:

$$L_{\nu,k}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{k}\right)^{\nu + sk}}{s!\Gamma(\nu + s + 1)\Gamma(2\nu + s + 1)\ldots\Gamma(k\nu - \nu + s + 1)} \tag{6}$$

The equation for U_{n-2} is

$$D^n U_{n-2} + nD^{n-1}U_{n-1} = e^z U_{n-2}$$

and it is readily seen that

$$U_{n-2} = C_{n-2}E_n(e^z) - nc_{n-1}\left[\frac{e^z}{(1!)^n} + \left(1 + \frac{1}{2}\right)\frac{e^{2z}}{(2!)^n} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{e^{3z}}{(3!)^n} + \ldots\right. \tag{7}$$

where c_{n-2} is another arbitrary constant. The functions U_{n-3}, $U_{n-4}, \ldots$ may be found step by step and U_0 will involve n arbitrary constants.

Our aim now is to satisfy the partial differential equation by a series of type

$$V = \sum_{s=0}^{n-1} H_s U_s(z)$$

in which H_2 is a homogeneous polynomial of degree s in $l_1, l_2, \ldots l_n$. It is readily seen that the partial differential equation will be satisfied if

$$D_m H_p = \binom{n}{m} H_{p-m} \qquad \begin{matrix} p = 0, 1, \ldots n - 1 \\ m = 1, 2, \ldots n - 1 \end{matrix}$$

where the operators, $D_1, D_2, D_3, \ldots$ are defined by the equations

$$D_1 = \sum \frac{\partial}{\partial l_s}$$

$$D_2 = \sum \frac{\partial^2}{\partial l_r \partial l_s} \qquad r \neq s$$

$$D_3 = \sum \frac{\partial^3}{\partial l_r \partial l_s \partial l_t} \qquad r \neq s = t$$

and H_q is supposed to be zero when q is negative. From these equations it follows that the polynomials H_p all satisfy the system of partial differential equations

$$\frac{D_m H}{\binom{n}{m}} = \frac{D_1^m H}{\binom{n}{1}^m} \qquad m = 1, 2, \ldots n - 1.$$

When $n = 3$ there is just one equation

$$\frac{\partial^2 H}{\partial l_1^2} + \frac{\partial^2 H}{\partial l_2^2} + \frac{\partial^2 H}{\partial l_3^2} = \frac{\partial^2 H}{\partial l_2 \partial l_3} + \frac{\partial^2 H}{\partial l_3 \partial l_1} + \frac{\partial^2 H}{\partial l_1 \partial l_2}$$

and the general solution is

$$H = F(l_1 + l_2 + l_3, l_1 + \omega l_2 + \omega^2 l_3) + G(l_1 + l_2 + l_3, l_1 + \omega^2 l_2 + \omega l_3)$$

where F and G are arbitrary functions of their arguments and $1 + \omega + \omega^2 = 0$.

The natural generalization of this for an arbitrary value of n is

$$H = \sum_{\sigma} F_{\sigma}(z, l_1 + \sigma l_2 + \sigma^2 l_3 + \ldots \sigma^{n-1} l_n)$$

where σ is a primitive root of the equation $x^n = 1$ and the summation extends over all the primitive roots. This form of solution is particularly useful because when H_{n-1} has been expressed in this form the corresponding expression for H_{n-m-1} is

$$H_{n-m-1} = \sum_{\sigma} \frac{\partial^m F_{\sigma}}{\partial z^m}.$$

The most interesting type of logarithmic solution of the partial differential equation is

$$V = E_n(e^z) \log \frac{x_2}{x_1}.$$

This solution may be generalized by writing $x_p - u_p(t)$ in place of x_p. Multiplying by $f(t)$, where $f(t)$ is an arbitrary analytic function and integrating round a contour enclosing just one root of $x_1 = u_1(t_1)$ and just one root of $x_2 = u_2(t_2)$, we have the solution

$$V = \frac{1}{2\pi i} \int E_n\{[x_1 - u_1(t)][x_2 - u_2(t)] \ldots [x_n - u_n(t)]\} f(t) dt \log \frac{x_2 - u_2(t)}{x_1 - u_1(t)}$$

$$= \int_{t_1}^{t_2} f(t) dt\, E_n\{ \qquad \text{''} \qquad \}$$

by Murphy's theorem.[6] When $n = 3$ this result may be used to obtain Chaundy's solutions and many others.

[1] L. Bianchi, *Rome. Acc. L. Rend.* (5) **4,** 8–18 (1895).

[2] O. Niccoletti, *Ibid.* (5) **4,** 330–337 (1895).

[3] F. Sibirani, *Annali di Matematica* (3) **28,** 1–34 (1918).

[4] T. W. Chaundy, *Proc. London Math. Soc.* (2) **21,** 214 (1923).

[5] The functions U_s are found by Sibirani.

[6] H. Bateman, *Bull. Am. Math. Soc.,* **39,** 118–123 (1923).